Discreteness and quasiresonances in weak turbulence of capillary waves

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A numerical study is presented which deals with the kinematics of quasiresonant energy transfer in a system of capillary waves with a discrete wave number space in a periodic box. For a given set of initially excited modes and a given level of resonance broadening, the modes of the system are partitioned into two classes, one active, the other forbidden. For very weak nonlinearity the active modes are very sparse. It is possible that this sparsity explains discrepancies between the values of the Kolmogorov constant measured in numerical simulations of weakly turbulent cascades and the theoretical values obtained from the continuum theory. There is a critical level of nonlinearity below which the set of active modes has finite radius in wave number space. In this regime, an energy cascade to dissipative scales may not be possible and the usual Kolmogorov spectrum predicted by the continuum theory not realized.

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I. INTRODUCTION

Weak turbulence theory (WTT) is concerned with the statistical description of ensembles of weakly interacting dispersive waves usually subjected to large scale forcing and small scale dissipation. Statistically steady solutions can be exactly found which carry a finite flux of energy from the forcing scale to the dissipation scale [1]. Energy transfer between scales is associated with the resonant interaction of groups of waves whose wave vectors all lie on certain resonant manifolds which thread the wave vector space of the system. The resulting steady state energy distributions are described by the Kolmogorov-Zakharov (KZ) spectra which have been observed in both experimental [2] and numerical [3] studies.

WTT is usually built under the assumption that the system under study is infinite in extent, statistically homogeneous and isotropic. Experimental investigations of wave phenomena, however, usually deal with bounded systems and numerical simulations usually assume periodic boundary conditions. In both these cases, the wave vector space of the system is a discrete lattice rather than a continuum. This is a potentially crucial distinction—particularly in view of Kartashova's proof [4] that the resonant manifolds of systems of waves with the $k^{-3/2}$ dispersion law, of which deep water capillary waves provide an example, are completely destroyed by any discretisation of the wave vectors. In numerical simulations of capillary wave turbulence, Pushkarev and Zakharov [5] (PZ) have reported very pronounced deviations from the $k^{-7/4}$ KZ spectrum predicted by WWT. At very low levels of nonlinearity, they found that their system failed to produce a cascade and all the energy accumulated in a collection of relatively low-$\vec{k}$ modes. They called the resulting distribution a ''wedding cake'' spectrum and suggested that the anomalous behavior was due to the destruction of the resonant manifolds caused by discreteness. They showed that the set of active modes in the discrete system is a small subset of the total number of possible modes.

In this paper we present a simple kinematic model of energy transfer in a nonlinear wave system with a discrete wave vector space. Our picture is based on the assumption that for a given level of nonlinearity, the system exhibits a characteristic degree of nonlinear resonance broadening which effectively thickens the resonance manifolds enough to get around Kartashova's theorem. For very low levels of nonlinearity, this thickening is no longer sufficient to maintain the integrity of the resonance manifolds and the effects of discreteness begin to play a role. The model gives a qualitative explanation for the ''wedding cake'' spectrum. In addition, we find that there is a critical level of nonlinearity above which the ''wedding cake'' spectrum ceases to exist and flux spectra carrying energy to small scales become possible. Our model suggests that above this critical level of nonlinearity, there is a regime where the active modes remain very sparse and arranged in such a way that energy transfer in wave vector space is very anisotropic. We conjecture that this ''spectral intermittency'' might manifest itself by modifying the value of the Kolmogorov constant associated with the angle-averaged energy spectrum. This hypothesis seems to be supported by the fact that PZ measured a value for the Kolmogorov constant [5] which was significantly lower than the theoretical value even in the regime where the $k^{-7/4}$ spectrum was well established.

II. KINEMATIC MODEL OF QUASI-RESONANCES

Consider a system of capillary waves. The dispersion relation for such waves is of the form

$$\omega_k \equiv \omega(|\mathbf{k}|) = \sqrt{\sigma} k^{3/2}, \tag{1}$$

where σ is the coefficient of surface tension. The nonlinear interactions in this system are predominantly three wave so the resonant manifolds are defined by the pair of equations

$$\omega_{k_1} \pm \omega_{k_2} - \omega_{k_3} = 0, \quad \mathbf{k}_1 \pm \mathbf{k}_2 - \mathbf{k}_3 = \mathbf{0}, \tag{2}$$

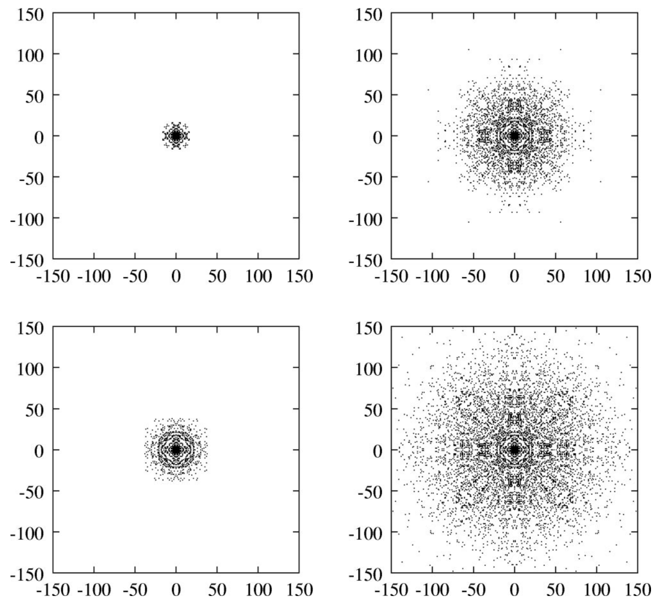

FIG. 1. Growth of the active region of k space for $\delta = 0.201\,746$. The successive generations have been grouped together. The four plots show generations 1-5, 6-10,11-15, and 16-19, respectively.

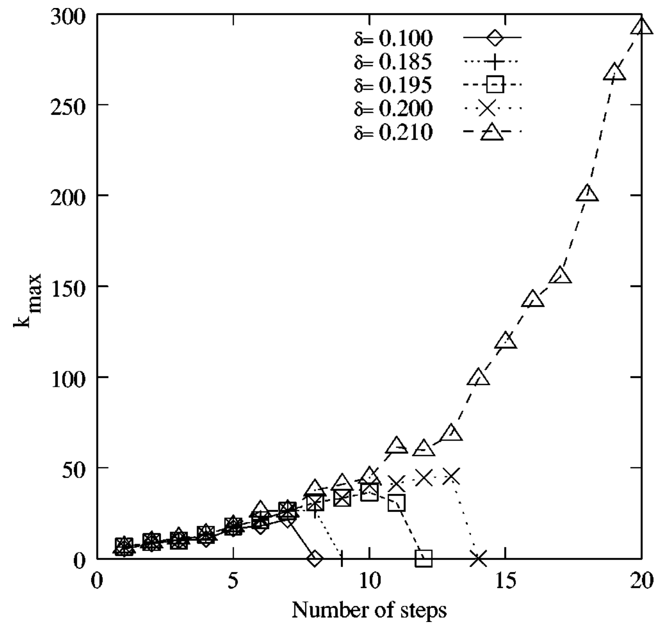

FIG. 2. Some cascades dying out.

with the $+$ sign corresponding to confluences, $1+2 \mapsto 3$, and the $-$ sign corresponding to decays, $1 \mapsto 2+3$. If the system is placed in a square box of side L then the values of $\mathbf{k}$ are quantized : $\mathbf{k} = \Delta k(n,m)$. Here $n,m \in \mathbb{Z}$ and $\Delta k = 2\pi/L$. As was shown by Kartashova [4], for integer valued vectors, the system of equations (2) has no solutions. The resonance conditions for nonlinear interactions cannot be satisfied in this case. The resolution of this apparent paradox lies in the fact that the dispersion relation (1) only holds exactly for linear waves. Once the nonlinear terms in the equations of motion are taken into account, the frequency, ω, acquires a weak dependence on the wave amplitude (see, e.g., Ref. [6] Chap. 14) This leads to a nonlinear correction to the equations describing the resonance manifolds

$$\omega_{k_1} \pm \omega_{k_2} - \omega_{k_3} = \Sigma, \quad \mathbf{k}_1 \pm \mathbf{k}_2 - \mathbf{k}_3 = \mathbf{0}. \tag{3}$$

Such interactions are called quasiresonances. The real part of Σ gives rise to nonlinear frequency shifts and the imaginary part gives rise to resonance broadening. We cannot compute Σ easily since it is functionally dependent on the entire spectrum. However we know that it must take a continuous range of values which we characterize by introducing a statistical characteristic level of resonance broadening, denoted by δ. If $|\Sigma| < \delta$, then the given combination of wave vectors can transfer energy. Therefore, as suggested in Ref. [3], we can model quasiresonant interactions by writing the resonance conditions as

$$|\omega_{k_1} \pm \omega_{k_2} - \omega_{k_3}| < \delta, \quad \mathbf{k}_1 \pm \mathbf{k}_2 - \mathbf{k}_3 = \mathbf{0}. \tag{4}$$

We effectively thicken the resonance manifolds by an amount of the order of the nonlinear resonance broadening. The characteristic level of broadening is related to the level of nonlinearity which is usually thought of in terms of the small parameter ϵ used to derive the weak turbulence perturbation expansion. For our purposes we do not need to know the details of this relationship. It is sufficient to remember that for weak nonlinearity, δ is small and $\delta \to 0$ in the linear limit, $\epsilon \to 0$.

We propose the following kinematic model to study the effects of discreteness and nonlinear broadening on the transfer of energy in the system.

(i) By rescaling the first of the quasiresonance conditions (4) by a factor of $\sqrt{\sigma}(\Delta k)^{3/2}$ we can write

$$|k_1^{3/2} \pm k_2^{3/2} - k_3^{3/2}| < \delta', \quad \mathbf{k}_1 \pm \mathbf{k}_2 - \mathbf{k}_3 = \mathbf{0}. \tag{5}$$

where δ' denotes the rescaling of the physical δ and the vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \in \mathbb{Z}^2$.

(ii) We put some energy into a small collection of initial modes. We denote this initial collection of excited modes by $\mathcal{S}_0$. Since we usually force at large scales, the modes in $\mathcal{S}_0$ are clustered around the origin in $\vec{k}$ space.

(iii) We now examine which modes can interact at the given level of nonlinear broadening. We construct a new set of modes as follows:

$$\mathcal{S}_1 = \{\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 : \mathbf{k}_1, \mathbf{k}_2 \in \mathcal{S}_0, \omega_{k_1} + \omega_{k_2} - \omega_k < \delta\}.$$

(iv) Define $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. Provided that δ is large enough, $\mathcal{S}_1$ will be nonempty and $\mathcal{S}$ will constitute larger set of possible active modes.

(v) We can now iterate this procedure to generate a series of cascade generations $\mathcal{S}_1, \ \mathcal{S}_2, \ldots, \mathcal{S}_N$. Their union gives us a map of the set of active modes in the system.

This model is purely kinematic. It does not say anything about how energy might be exchanged dynamically among the active modes. We shall see however that the kinematics alone allows one to make some interesting observations

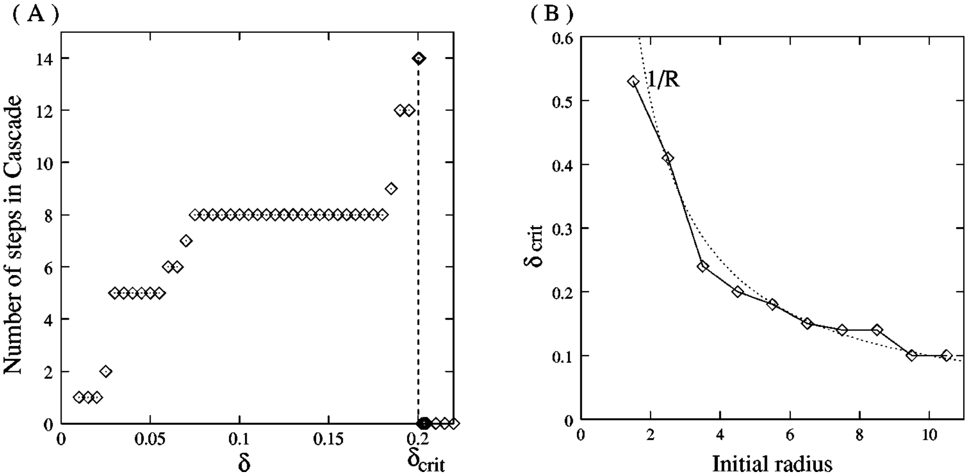

FIG. 3. (a) Total number of steps as a function of δ for initial forcing radius $K_0=4.5$. After $\delta_{\mathrm{crit}}=0.201\,746$ the number of steps is infinite. (b) Plot of δ_{crit} for a variety of values of the initial forcing radius K_0. The dotted line is the curve $1/K_0$.

about what happens at low levels of nonlinearity when δ is small and the weak turbulence begins to feel the effects of discreteness.

III. VERY WEAKLY NONLINEAR CASCADES

A. Behavior of the model for small δ

We take the initial set of excited modes to be the modes contained within a circle of radius K_0 about the origin. While this would be considered to be isotropic forcing in the continuum case, we must bear in mind that the discreteness of the lattice is felt very strongly when K_0 is small so it does not really make sense to talk about isotropic forcing in this case. Some maps of active modes are shown in Fig. 1. These plots are for $\delta=0.201\,746$ when the cascades are quite sparse. K_0 was taken, in this case, to be 4.5 which corresponds to an initial set of 69 excited modes.

$\delta=0$ corresponds to the linear picture where there is no exchange of energy between modes. In this case the energy just stays in the initial circle around 0. As δ is increased, it is clear that there exists $\delta_{\mathrm{min}}>0$, above which it becomes possible to satisfy the quasiresonance conditions (3). At this value of δ, cascades of energy to higher k become possible. Figure 2 shows the maximum modulus of the wave vectors in each successive generation of the cascades for a range of values of δ.

For small values of δ, the cascade proceeds for only a finite number of steps before dying out. Figure 3(a) shows the total number of steps in the cascade as a function of δ for initial forcing radius, $K_0=4.5$. The number of steps before extinction increases with δ until $\delta=\delta_{\mathrm{crit}}=0.201\,746\,2$ is reached whereupon the cascade suddenly escapes to infinity. It is also interesting to observe the quite extended plateaux in Fig. 3, particularly the range $0.075<\delta<0.018$, where the behavior of the model is quite insensitive to variations in the nonlinearity level.

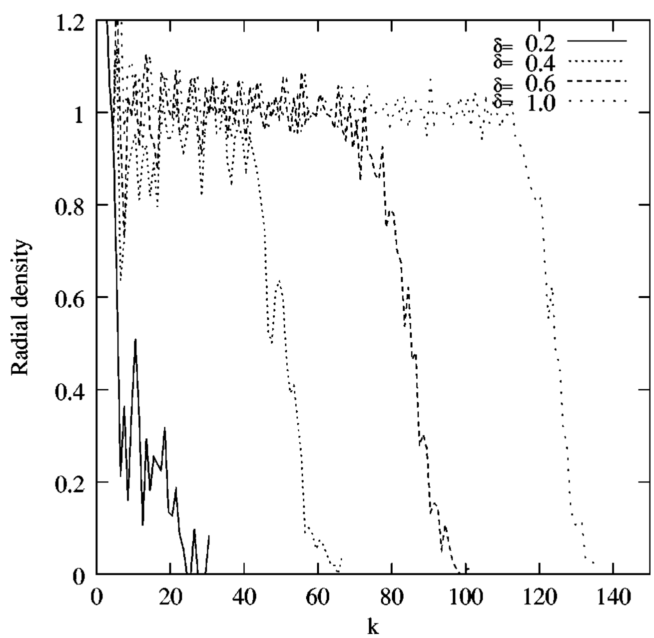

FIG. 4. Radial density of cascades for a range of values of δ.

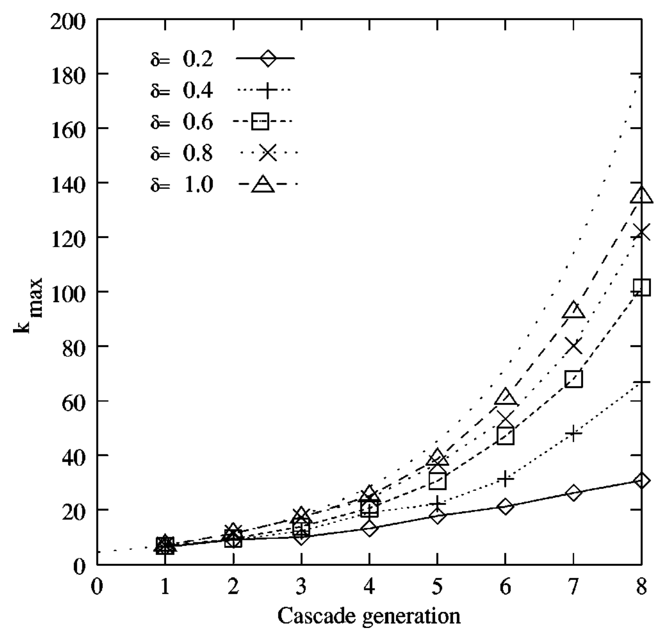

FIG. 5. Growth of cascades for a range of values of δ. The dotted curve is the continuum upper bound on k_{max} as a function of the cascade generation, $k_{\mathrm{max}}(n)\leqslant K_0 2^{2n/3}$.

The value $\delta = \delta_{\mathrm{crit}}$ appears to correspond to a sharp breakdown of the finite radius confinement. There does not appear to be a sequence of consecutively larger but finite cascades generated as $\delta \rightarrow \delta_{\mathrm{crit}}$. There is nothing special about the particular value of δ_{crit} since it depends on the initial modes chosen to start the cascade. Figure 3(b) shows the value of δ_{crit} obtained for different initial forcing characterized by the spectral radius of the forcing K_0. It was a surprise to find that $\delta_{\mathrm{crit}} \approx 1/K_0$ to quite a high accuracy. We do not see an obvious reason why this should be so.

B. Behavior for large δ

Figure 4 shows some plots of the angle averaged density of active modes as a function of k for different values of δ. As one might expect, as δ approaches 1 (which equals the lattice spacing in our model) the density approaches its continuum value of 1. The cascade growth rate also approaches the continuum value for large δ. In the continuum, one can easily show that the ratio of the radii of two successive steps in the cascade should be given by $k_{n+1}^{\mathrm{max}}/k_n^{\mathrm{max}} = 2^{2/3}$. Figure 5 shows that this cascade front speed is only beginning to be attained as δ approaches 1. For intermediate values, the cascade grows much more slowly. It is interesting, however, that the density of active modes in $\vec{k}$ space becomes of order one behind the front, even for relatively low values of δ, while in the same regime, the front speed is strongly inhibited. Qualitatively, this may explain the slowdown of energy flux and lower Kolmogorov constant observed by Pushkarev and Zakharov in numerical experiments [5].

IV. DISCUSSION

We believe that the behavior observed in our model provides a possible explanation for the anomalous behavior observed by Pushkarev and Zakharov in their numerical simulations. First let us consider what might happen if a numerical experiment is carried out with the characteristic nonlinearity small enough that the nonlinear resonance broadening is, on average, less than δ_{crit}. The energy initially supplied to the system begins to cascade to high k but the cascade dies out due to the above kinematic considerations and the energy remains trapped in a roughly circular region but with a very sparse and anisotropic distribution. Such trapping of energy and the strongly nonuniform energy distribution on $|\mathbf{k}|$ (see Fig. 4 for $\delta = 0.2$) agrees with PZ's observations of the ''wedding cake'' structure. How this energy distributes itself to produce the ''layers'' would require some dynamical information which is not present at this level.

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